Fractal Complex Dimensions and Zeta Functions, with Applications to Fractal Geometry

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Abstract: We will give some sample results from the new higher-dimensional theory of complex fractal dimensions developed jointly with Goran Radunovic and Darko Zubrinic in the research monograph (joint with these same co-authors), "Fractal Zeta Functions and Fractal Drums: Higher Dimensional Theory of Complex Dimensions" [2], published by Springer in February 2017 in the Springer Monographs in Mathematics series. We will also explain its connections with the earlier one-dimensional theory of complex dimensions developed, in particular, in the research monograph (by the speaker and M. van Frankenhuijsen) entitled "Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings" [1] (Springer Monographs in Mathematics, Springer, New York, 2013; 2nd rev. and enl. edn. of the 2006 edn.).

In particular, to an arbitrary compact subset A of the N-dimensional Euclidean space (or, more generally, to any relative fractal drum), we will associate new distance and tube zeta functions, as well as discuss their basic properties, including their holomorphic and meromorphic extensions, and the nature and distribution of their poles (or 'complex dimensions'). We will also show that the abscissa of convergence of each of these fractal zeta functions coincides with the upper box (or Minkowski) dimension of the underlying compact set A, and that the associated residues are intimately related to the (possibly suitably averaged) Minkowski content of A. Example of classical fractals and their complex dimensions will be provided.

Finally, if time permits, we will discuss and extend to any dimension the general definition of fractality proposed by the author (and M-vF) in their earlier work [1], as the presence of nonreal complex dimensions. We will also provide examples of "hyperfractals", for which the 'critical line' {Re(s)=D}, where D is the Minkowski dimension, is not only a natural boundary for the associated fractal zeta functions, but also consist entirely of singularities of those zeta functions.

Fractal tube formulas are obtained which enable us to express the intrinsic oscillations of fractal objects in terms of the underlying complex dimensions and the residues of the associated fractal zeta functions. Intuitively, the real parts of the complex dimensions correspond to the amplitudes of the associated "geometric waves", while their imaginary parts correspond to the frequencies of those waves. This is analogous to Riemann's explicit formula in analytic number theory, expressing the counting function of the primes in terms of the underlying zeros of the celebrated Riemann zeta function.

These results are used, in particular, to show the sharpness of an estimate obtained for the abscissa of meromorphic convergence of the spectral zeta functions of fractal drums. Furthermore, we will also briefly discuss recent joint results in which we obtain general fractal tube formulas in this context (that is, for compact subsets of Euclidean space or for relative fractal drums), expressed in terms of the underlying complex dimensions. We may close with a brief discussion of a few of the many open problems stated at the end of the aforementioned book, [1].

In a series of joint joint papers with Claire David [3-6], this general theory of complex dimensions has recently been significantly extended and applied to the Weierstrass nowhere differentiable function (and a large class of other fractal curves, including the Koch snowflake curve) in order to obtain a corresponding fractal tube formula expressed in terms of the underlying complex dimensions, as well as the associated fractal cohomology. However, in this talk, we will not have the time to expound about the latter work.